A NOTE ON PRICING IN INCOMPLETE MARKETS

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1. PRICING INCOMPLETENESS

1.1. The set-up. Consider a financial market which, at the very least, is equipped with the standard risk free asset, \( B_t \), which follows the dynamics \( dB_t = rB_t dt \). As a bit of novelty, suppose a financial practitioner introduces a contingent claim written on some \( n \)-dimensional stochastic process \( x_t = (x_{1,t}, x_{2,t}, \ldots, x_{n,t}) \). Importantly, these \( x_i \) are not assumed to be price processes of traded financial assets, but rather remain complete generic empirically observable phenomena such as the humidity level in Portland, Oregon, the average headcount in an elementary school classroom across America and so forth. The only thing we will assume is that \( x_t \) is governed by a stochastic differential equation of the form

\[
dx{i,t} = \alpha_i(t, x_t) dt + \sum_{j=1}^{d} \beta_{ij}(t, x_t) dW_j,t,
\]

where \( \beta_i \equiv (\beta_{i1}, \beta_{i2}, \ldots, \beta_{id}) \) and \( W_t \equiv (W_{1,t}, W_{2,t}, \ldots, W_{d,t}) \) is a standard \( d \)-dimensional \( \mathbb{P} \)-Wiener process. The task at hand will be to deduce the arbitrage free price process \( V_t = V(t, x_t) \) of this claim, which we imagine is specified to have terminal pay-off \( \Phi(x_T) \). As we shall see, it turns out to be impossible to say anything concrete about the price process of any particular claim: nonetheless, as we have \( d \) sources of randomness, we can introduce \( d \) arbitrarily priced distinct claims on \( x_t \) after which any auxiliary claim will have its price determined uniquely by the preexisting benchmarks. In other words, once enough claims have been specified to hedge the risky components, any new claim will necessarily have to satisfy a criterion of internal consistency in order to rule out arbitrage.

1.2. The Pricing PDE. Suppose we have fixed the prices of \( d \) claims written on the state process \( x_t \). Let \( \Phi_i(x_t) \) be the terminal pay-off of the \( i \)-th claim, and let \( F^i_t = F^i(t, x_t) \) be the corresponding price process (which we assume exists). Jointly, these claims form our benchmark for pricing purposes. Our task is to deduce the arbitrage free price on the \((d+1)\)-th claim, \( V \), above. To this end, we recall from Itô’s Lemma that

\[
dV_t = V_t [\mu_{V,t} dt + \sigma_{V,t}^\top dW_t],
\]

where

\[
\mu_{V,t} = V_t^{-1} \left\{ \frac{\partial V}{\partial t} + \alpha^\top \frac{\partial V}{\partial x} + \text{tr} \left[ \beta^\top \frac{\partial^2 V}{\partial x \partial x^\top} \beta \right] \right\},
\]

\[
\sigma_{V,t}^\top = V_t^{-1} \left\{ \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \beta_i^\top \right\},
\]

and \( \alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_n) \) is a \( n \)-vector and \( \beta \equiv (\beta_1^\top, \beta_2^\top, \ldots, \beta_n^\top) \) is an \( n \times d \) matrix.

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Obviously, analogous expressions hold for each of the $d$ additional claims $\{F^1_t, F^2_t, \ldots, F^d_t\}$. We denote the corresponding drift and diffusion coefficients by $\mu_{F,t}$ and $\sigma^T_{F,t}$ for $i = 1, 2, \ldots, d$.

Suppose now that we form a portfolio based on the $d + 1$ different risky assets at our disposal. The time $t$ value process of this portfolio is of the form

$$
\Pi_t = \sum_{i=1}^{d} h^F_i F^i_t + h^V_t V_t
$$

where $h^F_i$ represent the number of units held of the $i^{th}$ asset, and $h^V$ is the number of units held of asset $V$.

The portfolio is envisioned to be self-financing meaning that it obeys the dynamics

$$
d\Pi_t = \Pi_t \left\{ \sum_{i=1}^{d} w^F_i \frac{dF^i_t}{F^i_t} + w^V_t \frac{dV_t}{V_t} \right\}
$$

$$
= \Pi_t \left\{ \left( \sum_{i=1}^{d} w^F_i \mu_{F,t} + w^V_t \mu_{V,t} \right) dt + \left( \sum_{i=1}^{d} w^F_i \sigma^T_{F,t} + w^V_t \sigma^T_{V,t} \right) dW_t \right\},
$$

where $w^F_i \equiv h^F_i F^i_t / \Pi$ and $w^V \equiv h^V V_t / \Pi$ are weights which sum to unity. Now, since there are more assets than sources of risk, it should be possible to make the portfolio locally risk free: explicitly, there should be some combination of weights $\{w_1, w_2, \ldots, w_d, w_V\}$ such that

$$
\sum_{i=1}^{d} w^F_i \mu_{F,t} + w^V_t \mu_{V,t} = 0,
$$

and

$$
\sum_{i=1}^{d} w^F_i \sigma^T_{F,t} + w^V_t \sigma^T_{V,t} = 0^T.
$$

From a matrix perspective, this result may be restated as

$$
\begin{pmatrix}
[\mu_1, t - r] & [\mu_2, t - r] & \cdots & [\mu_d, t - r] & [\mu_{V,t} - r]
\end{pmatrix}
\begin{pmatrix}
\sigma^T_{F,t} \\
\sigma^T_{F,t} \\
\vdots \\
\sigma^T_{F,t} \\
\sigma^T_{V,t}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}.
$$

The advantage of this formulation is as follows: if the multiplying $(d+1) \times (d+1)$ matrix is invertible, then the unique solution to the system is $(w^F, w^2, \ldots, w^d, w_V) = (0, 0, 0, 0)$ which obviously is a contradiction. The rank of the matrix is therefore less than $d + 1$; in particular, it must be the case that the top row can be written as a linear combination of the bottom $d$ rows:

$$
\mu_{F,t} - r = \sigma^T_{F,t} \lambda_t, \quad (i = 1, 2, \ldots, d) \quad \text{and} \quad \mu_{V,t} - r = \sigma^T_{V,t} \lambda_t
$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$. More succinctly, the first $d$ equations may also be formulated as $\mu_{F,t} - r \mathbf{1} = \sigma^T_{F,t} \lambda$ where $\mu_{F} = (\mu^1, \mu^2, \ldots, \mu^d)$, $1 = (1, 1, \ldots, 1) \in \mathbb{R}^d$ and $\sigma_F$ is the $d \times d$ matrix $(\sigma^T_{F,1}, \sigma^T_{F,2}, \ldots, \sigma^T_{F,d})$.

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1To see why the drift coefficient takes on the form that it does, observe that

$$
\mu y = V_{-1} \left\{ \frac{\partial V}{\partial t} dt + \sum_{i=1}^{n} \alpha_i \frac{\partial V}{\partial x_i} dt + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{d} \sum_{q=1}^{d} \sigma_{ij,kq} \frac{\partial^2 V}{\partial x_i \partial x_j} \frac{dW_{s,t}}{\delta_{sq}} dt \right\}
$$

$$
= V_{-1} \left\{ \frac{\partial V}{\partial t} dt + \sum_{i=1}^{n} \alpha_i \frac{\partial V}{\partial x_i} dt + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{d} \sum_{q=1}^{d} \sigma_{ij,kq} \frac{\partial^2 V}{\partial x_i \partial x_j} \frac{dW_{s,t}}{\delta_{sq}} dt \right\}
$$

where $\delta_{sq}$ is the Kronecker delta (which is unity iff $s = q$, otherwise zero). Per definition, the trace of a square matrix is defined as the sum of the components along the leading diagonal. In particular, if $A \in \mathbb{R}^{d \times d}$ then $\text{tr}[A] = \sum_{i=1}^{d} A_{ii}$, if $B \in \mathbb{R}^{d \times n}$ and $C \in \mathbb{R}^{n \times d}$ then $\text{tr}[BC] = \sum_{i=1}^{d} \sum_{q=1}^{n} B_{iq} C_{q}$, and so forth.
The derivation of the governing PDE of $V$ is now apparent: combing the expressions in (1) with the formula
\[ \mu_{V,t} - r = \sigma_{V,t}^T \lambda_t \]
we obtain, after a few manipulations,
\[
\frac{\partial V}{\partial t} + \sum_{i=1}^n \left( \alpha_i - \sum_{j=1}^d \beta_{ij} \lambda_j \right) \frac{\partial V}{\partial x_i} + \text{tr} \left[ \beta_t^T \frac{\partial^2 V}{\partial x \partial x^T} \beta_t \right] = rV,
\]
(2)

The principle of no arbitrage and the preexisting prices of $d$ similar claims thus enforce a definite pricing PDE upon the claim of interest ($V$).

1.3. Aside: recovering the multi-dimensional BS PDE. To see how this blends in with conventional "complete market" theory, let us imagine that $d = n$ and that $x_t$ is, in fact, the price process of $n$ different tradeable assets ($S^1_t, S^2_t, ..., S^n_t$). Then $\alpha_i = \mu^i_S S^i_t$ and $\beta^i_t = S^i_t \sigma^i_t$ whence
\[
\alpha_i - \sum_{j=1}^n \beta_{ij} \lambda_j = \left( \mu^i_S - \sum_{j=1}^n \sigma^i_{S,j} \lambda_j \right) S^i_t
\]
\[= rS^i_t \]
where the last line makes use of the result that $\mu^i_{S,t} - r = \sigma^i_{S,t}^T \lambda_t$. Thus, we recover the well-known pricing PDE
\[
\frac{\partial V}{\partial t} + rS^t \frac{\partial V}{\partial S} + \text{tr} \left[ \{\text{diag}[S^t \sigma_t] \}^T \frac{\partial^2 V}{\partial S \partial S^T} \{\text{diag}[S^t \sigma_t] \} \right] = rV,
\]
(3)

where $S \equiv (S^1, S^2, ..., S^d)$. 