THE MERTON PROBLEM

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1. The Problem Posed

1.1. **The Framework.** We consider the case of an investor who is assumed to live over a known temporal horizon $[0,T]$. His total wealth, $\mathscr{W}_t$, is modelled dynamically in time by a stochastic differential equation and is assumed to have the known initial value $\mathscr{W}_0 = w_0$. At any given instant the investor is faced with the choice of how much of his wealth to consume, c_t, and which proportion of his wealth, π_t, he should allocate to a risky asset (which we assume follows 1D geometric Brownian motion, $dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}$, where μ and σ are known constants). The remaining wealth is to be placed (with proportion $1-\pi_t$) in a riskless asset, which grows at the constant rate of interest r. Furthermore, we assume that consumption is everywhere non-negative, $c_t \geq 0$, whilst no such condition is placed on π_t (that is to say, we allow short selling of assets). If u is the investors utility function, and δ is some subjective discount factor, then Merton's portfolio problem [1969] is to find functions $c_t^* = c^*(t,\mathscr{W}_t)$ and $\pi_t^* = \pi^*(t,\mathscr{W}_t)$, $t \in [0,T]$, such that

$$\mathcal{I}(\pi_t, c_t|t=0,w_0) \equiv \mathbb{E}^{\mathbb{P}}_{0,w_0}\left[\int_0^T e^{-\delta t}u(c_t)dt + e^{-\delta T}u(\mathscr{W}_T)\right] \tag{1}$$

is maximized. I.e. our aim is to find a consumption-investment strategy such that the expected discounted utility of consumption over a life-time and the expected discounted utility of the bequest $\mathscr{W}_T$ is at its peek. To this end, let us assume we operate with utility of the constant relative risk aversion (CRRA) variety $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ where γ codifies the investor's risk aversion.

Finally, assume the overall portfolio dynamics is self-financing and that there are no monetary injections such as labour income. From Björk's [2009] Lemma 6.4[1] it follows that the wealth dynamics is

$$d\mathscr{W}_t = \mathscr{W}_t\left(\pi_t\frac{dS_t}{S_t} + (1-\pi_t)\frac{dB_t}{B_t}\right) - c_t dt \tag{2}$$

where B_t is the risk free investment, meaning that $dB_t = rB_t dt$. Substituting in the dynamics of the assets and doing a slight rearrangement this becomes

[1] Here's the idea: let $\boldsymbol{S}_t \in \mathbb{R}^n$ be a pricing vector, and let $\boldsymbol{h}_t \in \mathbb{R}^n$ be the portfolio holding, such that the investor's total wealth at time t is $\mathscr{W}_t = \boldsymbol{h}_t^\intercal \boldsymbol{S}_t$. Suppose the investor last updated his portfolio at time $t-\Delta t$ (holding $\boldsymbol{h}_{t-\Delta t}$), then the value of his portfolio at t is $\mathscr{W}_t = \boldsymbol{h}_{t-\Delta t}^\intercal \boldsymbol{S}_t$. The cost of the new portfolio he buys at t is $\boldsymbol{h}_t^\intercal \boldsymbol{S}_t$. We allow for proceeds consumption of the magnitude $c_t\Delta t$ in the interval Δt i.e. all in all the self-financing condition is $\boldsymbol{h}_{t-\Delta t}^\intercal \boldsymbol{S}_t = \boldsymbol{h}_t^\intercal \boldsymbol{S}_t + c_t\Delta t$ or identically $\Delta \boldsymbol{h}_t^\intercal \boldsymbol{S}_t + c_t\Delta t = 0$. Adding and subtracting $\Delta \boldsymbol{h}_t^\intercal \boldsymbol{S}_{t-\Delta t}$ and letting $\Delta t \to 0$ we get the budget equation $\boldsymbol{S}_t^\intercal d\boldsymbol{h}_t + d\boldsymbol{S}_t^\intercal d\boldsymbol{h}_t + c_t dt = 0$. But applying Itô to $\mathscr{W}_t = \boldsymbol{h}_t^\intercal \boldsymbol{S}_t$ we get $d\mathscr{W}_t = \boldsymbol{h}_t^\intercal d\boldsymbol{S}_t + \boldsymbol{S}_t^\intercal d\boldsymbol{h}_t + d\boldsymbol{S}_t^\intercal d\boldsymbol{h}_t$, which combined with our budget constraint gives us the self-financing condition $d\mathscr{W}_t = \boldsymbol{h}_t^\intercal d\boldsymbol{S}_t - c_t dt$ or identically $d\mathscr{W}_t = \mathscr{W}_t \sum_i \pi_{it} dS_{it}/S_{it} - c_t dt$ where we have defined the weight $\pi_{it} \equiv S_{it}h_{it}/\mathscr{W}_t$. Clearly, $\sum_i \pi_{it} = 1$ so the nomenclature 'weight' is appropriate.

(3) $$d\mathscr{W}_t = \pi_t[\mu - r]\mathscr{W}_t dt + (r\mathscr{W}_t - c_t)dt + \pi_t \sigma \mathscr{W}_t dW_t^{\mathbb{P}}.$$

1.2. **Remark.** From a purely axiomatic perspective, Merton's problem of utility optimization is a cacophony of dubious and overtly simplified assumptions not easily squared with real life investment-consumption processes of rational agents. Pitfalls include the (i) highly unrealistic two asset economy, (ii) the negligence of labour income and (iii) transactions costs, (iv) the constancy of δ, μ, σ and r, (v) the fixed lifetime of the investor and (vi) his simplified utility function. Nevertheless, we may rejoice in the fact that the relatively complex mathematical machinery of the problem above admits analytical solutions. Indeed, there is some solace to be sought in the more recent developments of the problem, which has addressed (most) of these issues and more.

2. The Hamilton-Jacobi-Bellman Equation

2.1. **The Derivation.** To solve the Merton problem (1), we must venture into the field of dynamical programming and solve the **Hamilton-Jacobi-Bellman (HJB) equation**. To this end, Munk [2008] offers a pedagogically excellent albeit perhaps mathematically unsophisticated argument from discrete to continuous time, whilst Øksendal [2003] provides a rigorous but also conceptually opaque account of the subject. We shall opt for an approach which very much follows the lines of Björk, hopefully striking a balance between the readily comprehensible and something from which the reader can also abstract a grander perspective.

The fundamental idea is to scrutinize the dynamics (the governing PDE) of the **optimal value function** (or **indirect utility function**) $\mathcal{V}(s, w_s) : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$, defined as

(4) $$\mathcal{V}(s, w_s) \equiv \sup_{\{\pi_t, c_t\}_{t \in [s,T]}} \mathcal{I}(\pi_t, c_t | s, w_s), \qquad \text{where}$$

$$\mathcal{I}(\pi_t, c_t | s, w_s) \equiv \mathbb{E}^{\mathbb{P}}_{s,w_s} \left[\int_s^T e^{-\delta t} u(c_t) dt + e^{-\delta T} u(\mathscr{W}_T) \right],$$

which, of course, is nothing but our original problem with a generic starting point $s \in [0, T]$ and wealth w_s (think of (4) as the scenario where we have to solve the Merton problem for an investor who has already lived for $s - t_0$ years). To accomplish this, we must first of all assume that

1. There are optimal functions $c_t^* : [s, T] \times \mathbb{R} \mapsto \mathbb{R}_0^+$ and $\pi_t^* : [s, T] \times \mathbb{R} \mapsto \mathbb{R}$ such that the supremum is attained, i.e. s.t. $\mathcal{V}(s, w_s) = \mathcal{I}(\pi_t^*, c_t^* | s, w_s)$. We say that there is an **optimal control law**, $\mathscr{L}^* : \{c_t^*, \pi_t^*\}$. This is an existence claim, but it is not a uniqueness claim.
2. $\mathcal{V} \in C^{1,2}$. In words, the first order temporal derivative, and the first and second order wealth derivatives of $\mathcal{V}$ all exist.
3. A number of limiting procedures in the following arguments can be justified.

Given these assumptions, the PDE can be derived by following these standard steps in dynamic programming:

1. Fix the coordinate $(s, w_s) \in [0, T] \times \mathbb{R}$ and consider the following two strategies over the interval $[s, T]$: **Strategy I** use the optimal control law $\mathscr{L}^* : \{\pi_t^*, c_t^*\}$. **Strategy II** Use the (sub)-optimal control law $\mathscr{L}' : \{\pi_t', c_t'\}$ where

$$\mathscr{L}' : \{\pi_t', c_t'\} \equiv \begin{cases} \mathscr{L} : \{\pi_t, c_t\}, & \text{for } (t, \mathscr{W}_t) \in [s, s + \Delta s] \times \mathbb{R} \\ \mathscr{L}^* : \{\pi_t^*, c_t^*\}, & \text{for } (t, \mathscr{W}_t) \in (s + \Delta s, T] \times \mathbb{R}, \end{cases}$$

where Δs is some incremental time step. Notice that it the optimal control is used over the latter time interval $(s+\Delta s, T]$.

(2) Compute the Merton expectation
$$\mathcal{I}(\pi_t, c_t|s, w_s) \equiv \mathbb{E}^{\mathbb{P}}_{s,w_s}\left[\int_s^T e^{-\delta t}u(c_t)dt + e^{-\delta T}u(\mathscr{W}_T)\right]$$
for both strategies.

(3) Evidently, strategy I has to be at least as good as strategy II vis-a-vis the Merton expectation. Using this, and letting $\Delta s \to 0$ we obtain the HJB PDE.

From assumption (1) the first strategy is trivially $\mathcal{I}(\pi^*_t, c^*_t|s, w_s) = \mathcal{V}(s, w_s)$. For the second strategy we observe that we switch from a random control ($\mathscr{L}$) to an optimal control ($\mathscr{L}^*$) after Δs amounts of time. The wealth will therefore evolve to the stochastic state $\mathscr{W}^{\mathscr{L}}_{s+\Delta s}$ at $s+\Delta s$ and thence to its terminal value $\mathscr{W}^{\mathscr{L}^*}_T$ at T. Thus,

$$\begin{aligned}
\mathcal{I}(\pi'_t, c'_t|s, w_s) =& \mathbb{E}^{\mathbb{P}}_{s,w_s}\left[\int_s^{s+\Delta s} e^{-\delta t}u(c_t)dt + \int_{s+\Delta s}^T e^{-\delta t}u(c^*_t)dt + e^{-\delta T}u(\mathscr{W}^{\mathscr{L}^*}_T)\right]\\
=& \mathbb{E}^{\mathbb{P}}_{s,w_s}\left[\int_s^{s+\Delta s} e^{-\delta t}u(c_t)dt + \mathbb{E}^{\mathbb{P}}_{s+\Delta s,\mathscr{W}^{\mathscr{L}}_{s+\Delta s}}\left[\int_{s+\Delta s}^T e^{-\delta t}u(c^*_t)dt + e^{-\delta T}u(\mathscr{W}^{\mathscr{L}^*}_T)\right]\right]\\
=& \mathbb{E}^{\mathbb{P}}_{s,w_s}\left[\int_s^{s+\Delta s} e^{-\delta t}u(c_t)dt + \mathcal{V}(s+\Delta s, \mathscr{W}^{\mathscr{L}}_{s+\Delta s})\right]
\end{aligned}$$

where the second equality uses the Law of Iterated Expectations, and the third equality the definition of the optimal value function. Hence, using the first insight from step (3) we have that strategies I and II compare as

$$\mathcal{V}(s, w_s) \geq \mathbb{E}^{\mathbb{P}}_{s,w_s}\left[\int_s^{s+\Delta s} e^{-\delta t}u(c_t)dt + \mathcal{V}(s+\Delta s, \mathscr{W}^{\mathscr{L}}_{s+\Delta s})\right]. \tag{5}$$

Now using assumption (2) we can use Itô's formula to write

$$\begin{aligned}
\mathcal{V}(s+\Delta s, \mathscr{W}^{\mathscr{L}}_{s+\Delta s}) =& \mathcal{V}(s, w_s) + \int_s^{s+\Delta s}\Big\{\partial_s\mathcal{V}(t, \mathscr{W}^{\mathscr{L}}_t)dt\\
&+ \partial_w\mathcal{V}(t, \mathscr{W}^{\mathscr{L}}_t)d\mathscr{W}^{\mathscr{L}}_t + \frac{1}{2!}\partial_{ww}\mathcal{V}(t, \mathscr{W}^{\mathscr{L}}_t)(d\mathscr{W}^{\mathscr{L}}_t)^2\Big\}
\end{aligned} \tag{6}$$

which combined with our wealth dynamics (3) becomes

$$\begin{aligned}
\mathcal{V}(s+\Delta s, \mathscr{W}^{\mathscr{L}}_{s+\Delta s}) =& \mathcal{V}(s, w_s) + \int_s^{s+\Delta s}\Big\{\partial_s\mathcal{V}(t, \mathscr{W}^{\mathscr{L}}_t) + \hat{\mathbf{A}}^{\mathscr{L}}\mathcal{V}(t, \mathscr{W}^{\mathscr{L}}_t)\Big\}dt\\
&+ \int_s^{s+\Delta s}\pi_t\sigma\mathscr{W}^{\mathscr{L}}_t\partial_w\mathcal{V}(t, \mathscr{W}^{\mathscr{L}}_t)dW^{\mathbb{P}}_t,
\end{aligned} \tag{7}$$

where we have defined the differential operator

$$\hat{\mathbf{A}}^{\mathscr{L}} \equiv \pi_t[\mu - r]\mathscr{W}_t^{\mathscr{L}}\partial_w + (r\mathscr{W}_t^{\mathscr{L}} - c_t)\partial_w + \frac{1}{2!}\pi_t^2\sigma^2(\mathscr{W}_t^{\mathscr{L}})^2\partial_{ww}^2. \tag{8}$$

Substituting (7) into inequality (5) and assuming sufficient integrability[2] in order for the stochastic integral to vanish, we obtain

$$0 \geq \mathbb{E}_{s,w_s}^{\mathbb{P}}\left[\int_s^{s+\Delta s}\{e^{-\delta t}u(c_t) + \partial_s\mathcal{V}(t,\mathscr{W}_t^{\mathscr{L}}) + \hat{\mathbf{A}}^{\mathscr{L}}\mathcal{V}(t,\mathscr{W}_t^{\mathscr{L}})\}dt\right]. \tag{9}$$

Suppose now we divide through on both sides by Δs and take the limit as $\Delta s \to 0$. If our expression exhibits sufficient regularity we can justify interchanging the limit and the expectation operator. Thus,

$$0 \geq e^{-\delta s}u(c_s) + \partial_s\mathcal{V}(s,w_s) + \hat{\mathbf{A}}^{\mathscr{L}}\mathcal{V}(s,w_s), \tag{10}$$

Notice that our functions $\mathcal{V}, \pi_t, c_t$ (and consequently also $\hat{\mathbf{A}}^{\mathscr{L}}$) here are evaluated at the initial coordinate (s, w_s). However, whilst (s, w_s) hitherto has been treated as a fixed, it was arbitrarily chosen and thence equation (10) must hold true for all $(s, w_s) \in [0, T] \times \mathbb{R}$, with equality holding for the optimal control $\mathscr{L}^*$ only. Hence, we arrive at the theorem:

Theorem 1. The Hamilton-Jacobi-Bellman Equation for Merton's Problem. Consider a wealth process (3). Let $\mathcal{V}(s, w_s)$ be defined as in (4), and assume it satisfies assumptions (1)-(3) declared above, then $\mathcal{V}(s, w_s)$ satisfies the HJB equation

$$0 = \partial_s\mathcal{V}(s,w_s) + \sup_{c_s\in\mathbb{R}_0^+,\pi_s\in\mathbb{R}}\left\{e^{-\delta s}u(c_s) + \hat{\mathbf{A}}^{\mathscr{L}}\mathcal{V}(s,w_s)\right\}, \tag{11}$$

$\forall(s, w_s) \in (t_0, T) \times \mathbb{R}$, where

$$\hat{\mathbf{A}}^{\mathscr{L}} \equiv \pi_s[\mu - r]w_s\partial_w + (rw_s - c_s)\partial_w + \frac{1}{2!}\pi_s^2\sigma^2 w_s^2\partial_{ww}^2$$

and we have the obvious boundary condition $\mathcal{V}(T, w_T) = e^{-\delta T}u(w_T), \forall w_T \in \mathbb{R}$ (if we start the Merton problem when the investor dies there's nothing but the bequest). For each $(s, w_s) \in [0, T] \times \mathbb{R}$ the supremum is attained by c_s^*, π_s^*.

Remark. Importantly, the HJB equation (11), whilst highly non-linear, "only" involves the supremum over all admissible consumptions and holdings of risky assets at time s, and *not* the supremum over the entire process as we saw it in (4).

Do notice that the theorem above only has the form of a necessary condition: i.e. if $\mathcal{V}$ is an optimal value function and $\mathscr{L}^*$ an optimal control, then $\mathcal{V}$ satisfies the HJB equation with $\mathscr{L}^*$ giving rise to the supremum. For computational purposes even more interesting is the fact the HJB equation per se serves as a sufficient condition for the optimal control problem. This idea is captured in the so-called *verification theorem* which decrees:

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[2]In practice this translates to the condition that $\pi_t^2(\mathscr{W}_t^{\mathscr{L}})^2\partial_{ww}^2\mathcal{V} \in \mathcal{L}^2$. For recall that square integrable functions $g \in \mathcal{L}^2[a, b]$ have the property that $\mathbb{E}_a\left[\int_a^b g(u)dW_u\right] = 0$ cf. Björk's Proposition 4.7.

Theorem 2. The Verification Theorem for Merton's Problem. Suppose we have the functions $\mathcal{H}(s, w_s)$, $\pi^*(s, w_s)$ and $c^*(s, w_s)$ such that

- $\mathcal{H}$ is sufficiently integrable (see footnote 2) and solves the HJB equation
$$0 = \partial_s \mathcal{H}(s, w_s) + \sup_{c_s \in \mathbb{R}_0^+, \pi_s \in \mathbb{R}} \left\{ e^{-\delta s} u(c_s) + \hat{\mathbf{A}}^{\mathscr{L}} \mathcal{H}(s, w_s) \right\},$$
$\forall (s, w_s) \in (0, T) \times \mathbb{R}$, with the terminal condition $\mathcal{H}(T, w_T) = e^{-\delta T} u(w_T), \forall w_T \in \mathbb{R}$.
- $\pi^*(s, w_s) : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ and $c^*(s, w_s) : [0, T] \times \mathbb{R} \mapsto \mathbb{R}_0^+$ - that is, π^* and c^* are admissible control laws (they satisfy the pre-specified function constraints).
- For each fixed (s, w_s) the supremum in the expression
$$\sup_{c_s \in \mathbb{R}_0^+, \pi_s \in \mathbb{R}} \left\{ e^{-\delta s} u(c_s) + \hat{\mathbf{A}}^{\mathscr{L}} \mathcal{H}(s, w_s) \right\}$$
is attained by the choice $\pi_s = \pi^*(s, w_s)$, $c_s = c^*(s, w_s)$.

Then it holds that

(1) The optimal value function (4) to Merton's control problem is given by $\mathcal{V}(s, w_s) = \mathcal{H}(s, w_s)$.
(2) There exist an optimal control law, viz. $\{\pi^*(s, w_s), c^*(s, w_s)\}$.

Proof. Let functions $\mathcal{H}, \pi^*$ and c^* be given as above. Select the arbitrary admissible control law $\mathcal{L} : \{\pi_t, c_t\}$ and fix a coordinate (s, w_s). If we define the dynamics of the wealth process $\mathscr{W}_t^{\mathscr{L}}$ as in (3) with boundary $\mathscr{W}_s^{\mathscr{L}} = w_s$, then an straight-forward application of Itô implies that

$$\begin{aligned} \mathcal{H}(T, \mathscr{W}_T^{\mathscr{L}}) = \mathcal{H}(s, w_s) + \int_s^T \left\{ \partial_s \mathcal{H}(t, \mathscr{W}_t^{\mathscr{L}}) + \hat{\mathbf{A}}^{\mathscr{L}} \mathcal{H}(t, \mathscr{W}_t^{\mathscr{L}}) \right\} dt \\ + \int_s^T \pi_t \sigma \mathscr{W}_t \partial_w \mathcal{H}(t, \mathscr{W}_t^{\mathscr{L}}) dW_t^{\mathbb{P}}. \end{aligned} \tag{12}$$

Using our assumptions that $\mathcal{H}$ satisfies the HJB equation and has the terminal value $\mathcal{H}(T, \mathscr{W}_T^{\mathscr{L}}) = e^{-\delta T} u(\mathscr{W}_T^{\mathscr{L}})$ we get

$$\mathcal{H}(s, w_s) \geq \int_s^T e^{-\delta t} u(c_t) dt + e^{-\delta T} u(\mathscr{W}_T^{\mathscr{L}}) - \int_s^T \pi_t \sigma \mathscr{W}_t \partial_w \mathcal{H}(t, \mathscr{W}_t^{\mathscr{L}}) dW_t^{\mathbb{P}}. \tag{13}$$

Applying the (t, w_s) conditional expectation to this equation, and using the integrability assumption:

$$\mathcal{H}(s, w_s) \geq \mathbb{E}_{t, w_s}^{\mathbb{P}} \left[\int_s^T e^{-\delta t} u(c_t) dt + e^{-\delta T} u(\mathscr{W}_T^{\mathscr{L}}) \right] \equiv \mathcal{I}(\pi_t, c_t | s, w_s). \tag{14}$$

This inequality is true for arbitrary control laws - also in the event that we selected the supremal control law. Hence, from the definition of $\mathcal{V}$, (4):

$$\mathcal{H}(s, w_s) \geq \mathcal{V}(s, w_s). \tag{15}$$

Had we opted for using the functions π^*, c^* it is clear that we would have obtained a strict equality in equation (14) viz. $\mathcal{H}(s, w_s) = \mathcal{I}(\pi_t^*, c_t^* | s, w_s)$. If we substitute this into the trivial inequality $\mathcal{V}(s, w_s) \geq \mathcal{I}(\pi_t^*, c_t^* | s, w_s)$ we get:

$$\mathcal{V}(s, w_s) \geq \mathcal{H}(s, w_s). \tag{16}$$

Evidently, (15) and (16) jointly imply $\mathcal{H} = \mathcal{V}$ and that $\{\pi_t^*, c_t^*\}$ is an optimal control. □

3. Solving Merton's Problem

3.1. **Is the HJB equation solvable?** Qua the inherent non-linearity of the HJB equation, the reader might reasonably ask whether we have made any significant progress regarding solvability of the optimal value function? The answer is one of ambivalence: it will hardly come as a surprise that the Merton problem in particular admits nice analytic solutions. Nonetheless, there are analogous optimization problems with analogous HJB equations that fare less well in this respect.

The protocol we follow when searching for a solution to the HJB equation is roughly as follows. First of all we fix an arbitrary coordinate in time and wealth space and find the control functions (π, c) for which the expression under the supremum sign attains its maximum. This is a matter of straightforward differentiation. However, these controls will naturally depends on the (as of yet unknown) function $\mathcal{V}$ and its various derivatives. Next, based on the terminal condition, we make an **ansatz** as to the general form of $\mathcal{V}$, which in turn typically involves some unknown function f. Plug this ansatz into the partial derivatives and our control functions, and then substitute these equations into the HJB PDE (now with strict equality as we are using a posited optimal control law). If we are fortunate the resulting differential equation in f will be solvable.

3.2. **The Solution.** Written explicitly, the equation we need to solve is of the form

$$0 = \partial_s \mathcal{V} + \sup_{c_s \geq 0, \pi_s} \left\{ e^{-\delta s} \frac{c_s^{1-\gamma}}{1-\gamma} + \pi_s[\mu - r] w_s \partial_w \mathcal{V} + (r w_s - c_s) \partial_w \mathcal{V} + \frac{1}{2!} \pi_s^2 \sigma^2 w_s^2 \partial_{ww}^2 \mathcal{V} \right\},$$

$$\text{s.t. } \mathcal{V}(T, w_T) = e^{-\delta T} \frac{w_T^{1-\gamma}}{1-\gamma}.$$

The initial static optimization problem to be solved is trivial. We must simply differentiate the $\{\ \}$-expression with respect to c_s and π_s and equate to zero in order to get the **first order conditions**:

$$\partial_c \{\ \} = 0 : e^{-\delta s} c_s^{\gamma} - \partial_w \mathcal{V} = 0 \Leftrightarrow c_s = \left(e^{\delta s} \partial_w \mathcal{V}\right)^{-\frac{1}{\gamma}} \tag{17a}$$

$$\partial_w \{\ \} = 0 : [\mu - r] w_s \partial_w \mathcal{V} + \pi_s \sigma^2 w_s^2 \partial_{ww}^2 \mathcal{V} = 0 \Leftrightarrow \pi_s = \frac{-[\mu - r] \partial_w \mathcal{V}}{\sigma^2 w_s \partial_{ww}^2 \mathcal{V}} \tag{17b}$$

Clearly, our controls depends on $\mathcal{V}$. We therefore make the ansatz that the solution is of the form

$$\mathcal{V}(s, w_s) = e^{-\delta s} \frac{w_s^{1-\gamma}}{1-\gamma} f(s), \tag{18}$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is a function which obeys $f(T) = 1$ cf. the terminal condition. Differentiating this expression wrt s, w and ww we obtain

$$\partial_s \mathcal{V} = e^{-\delta s}\frac{w_s^{1-\gamma}}{1-\gamma}\dot{f}(s) - \delta e^{-\delta s}\frac{w_s^{1-\gamma}}{1-\gamma}f(s) \tag{19a}$$

$$\partial_w \mathcal{V} = e^{-\delta s} w_s^{-\gamma} f(s) \tag{19b}$$

$$\partial_{ww}^2 \mathcal{V} = -\gamma e^{-\delta s} w_s^{-\gamma-1} f(s) \tag{19c}$$

where the dot above the f symbolizes the strict temporal derivative. Substituting (19) back into (17) we obtain the elegant results

$$\pi_s^* = \frac{\mu - r}{\sigma^2\gamma}, \tag{20a}$$

$$c_s^* = w_s f(s)^{-\frac{1}{\gamma}}, \tag{20b}$$

but, of course, we have yet to find out which dynamics governs f. As suggested above, there's only one way to find out: plug (19) and (20) into the HJB equation (3.2). Inevitably, this becomes rather messy in the beginning, but it simplifies considerably by eliminating common factors and relabeling constants:

$$\begin{aligned}
0 =& e^{-\delta s}\frac{w_s^{1-\gamma}}{1-\gamma}\dot{f}(s) - \delta e^{-\delta s}\frac{w_s^{1-\gamma}}{1-\gamma}f(s) + \Bigg\{ e^{-\delta s}\frac{w_s^{1-\gamma}}{1-\gamma}f(s)^{\frac{\gamma-1}{\gamma}} \\
&+ \frac{[\mu-r]^2}{\sigma^2\gamma} w_s e^{-\delta s} w^{-\gamma} f(s) + \left(r w_s - w_s f(s)^{-\frac{1}{\gamma}} \right) e^{-\delta s} w_s^{-\gamma} f(s) \\
&- \frac{1}{2!}\frac{[\mu-r]^2}{\sigma^4\gamma^2}\sigma^2 w_s^2 \gamma e^{-\delta s} w_s^{-\gamma-1} f(s) \Bigg\} \\
\Leftrightarrow 0 =& w_s^{1-\gamma}\Bigg[\frac{1}{1-\gamma}\dot{f}(s) - \frac{\delta}{1-\gamma}f(s) + \frac{1}{1-\gamma}f(s)^{\frac{\gamma-1}{\gamma}} + \frac{[\mu-r]^2}{\sigma^2\gamma}f(s) \\
&+ r f(s) - f(s)^{\frac{\gamma-1}{\gamma}} - \frac{1}{2}\frac{[\mu-r]^2}{\sigma^2\gamma}f(s) \Bigg] \\
\Leftrightarrow 0 =& w_s^{1-\gamma}\Bigg[\dot{f}(s) + \left(\frac{1}{2}\frac{[\mu-r]^2(1-\gamma)}{\sigma^2\gamma} + r(1-\gamma) - \delta \right) f(s) + \gamma f(s)^{\frac{\gamma-1}{\gamma}} \Bigg].
\end{aligned}$$

Defining the constant

$$\Gamma \equiv \frac{1}{2}\frac{[\mu-r]^2(1-\gamma)}{\sigma^2\gamma^2} + r\frac{1-\gamma}{\gamma} - \frac{\delta}{\gamma} \tag{21}$$

and using the fact that our equation must hold for all w_s and all s we reduce our expression to

$$0 = \dot{f}(s) + \gamma\Gamma f(s) + \gamma f(s)^{\frac{\gamma-1}{\gamma}}, \quad \text{s.t. } f(T) = 1. \tag{22}$$

This is a **Bernoulli equation** and it can easily be solved. The trick is to define a function

$$g(s) \equiv f(s)^{\frac{1}{\gamma}} \tag{23}$$

such that $f(s) = g(s)^\gamma$ and $\dot{f}(s) = \gamma \dot{g}(s) g(s)^{\gamma-1}$. Plugging this into (22) it is easily to verify that

$$0 = \dot{g}(s) + \Gamma g(s) + 1, \quad \text{s.t. } g(T) = 1. \tag{24}$$

Using the integration multiplier $e^{\Gamma t}$ we can write this as equation as

$$0 = \frac{d}{ds}[e^{\Gamma s} g(s)] + e^{\Gamma s}$$

which integrates to

$$0 = e^{\Gamma s} g(s) + \Gamma^{-1} e^{\Gamma s} + C$$

where C is a constant. Since $g(T) = 1$ we find that

$$C = -(1 + \Gamma^{-1}) e^{\Gamma T}$$

and thence

$$g(s) = (1 + \Gamma^{-1}) e^{\Gamma(T-s)} - \Gamma^{-1}. \tag{25}$$

Collecting all of these results, we finally arrive at:

$$\pi_s^* = \frac{\mu - r}{\sigma^2 \gamma}, \tag{26a}$$

$$c_s^* = \frac{w_s \Gamma}{(1+\Gamma) e^{\Gamma(T-s)} - 1}, \tag{26b}$$

$$\mathcal{V}(s, w_s) = e^{-\delta s} \frac{w_s^{1-\gamma}}{1-\gamma} \left((1 + \Gamma^{-1}) e^{\Gamma(T-s)} - \Gamma^{-1} \right)^{\gamma}. \tag{26c}$$

3.3. **Economic Analysis.** The optimal proportion allocated to the risky asset, (26a), exhibits independence of time and wealth, being proportional to excess return and inversely proportional to variance of returns and risk aversion. From a modern portfolio theoretic perspective this is in accordance with our expectations. Too see this consider the simple two asset economy where our expected utility of portfolio returns is of the mean-variance form $\mathbb{E}[U(r_p)] = \mathbb{E}[r_p] - \frac{\gamma}{2}\mathbb{V}\text{ar}[r_p]$. Since our entire wealth is divided between a risky and a risk free asset, the return on our portfolio is $r_p = \pi r_r + (1-\pi) r_f$ where $\mathbb{E}[r_r] = \mu$ and $\mathbb{V}\text{ar}[r_r] = \sigma^2$. Thus, $\mathbb{E}[U(r_p)] = \pi\mu + (1-\pi) r_f - \frac{\gamma}{2}\pi^2\sigma^2$, which readily is seen to be maximized when $\pi^* = \frac{\mu - r_f}{\sigma^2 \gamma}$.

For optimal consumption (26b) consider the empirically plausible case of $\gamma \approx 1$[3]. Then $\Gamma \approx -\delta$ (the subjective discount factor of the agent) and

$$c_s^* \approx \frac{w_s \delta}{1 + (\delta - 1) e^{-\delta(T-s)}}.$$

This means that optimal consumption is linear in the wealth variable, but decreases exponentially in time, which is certainly plausible.

[3] Rolf Poulsen would say that γ is closer to 2-5.