1. The Currency Derivative Pricing Differential Equation

1.1. Overview. The nomenclature currency derivative is used to refer to a broad class of contingent claims, which are unified by the fact that they somehow depend on an exchange rate between two markets (conveniently labelled the domestic and the foreign market - the former being the home turf of the option holder). This exchange rate might be agreed upon in advance in which case we would expect the option to depend on at least one other stochastic process such as a foreign equity security. An example of this would be a quanto put, which gives the holder the right, but not the obligation, to sell one unit of a foreign stock at a predetermined time in the future at a predetermined strike and receive the pay-off at a prearranged exchange rate. However, the option may also be written on nothing but a stochastically varying exchange rate as exemplified by currency forwards, which lock in the exchange rate at which some notional amount can be acquired in the future.

To launch our study of currency options we shall make a number of simplifying assumptions: first, we assume that both the domestic and foreign markets are frictionless and liquid. Secondly, the prevailing interest rates at which money is borrowed or lent in these markets are constant but not necessarily equal. We label them \( r_d \) and \( r_f \), where \( d \) and \( f \) henceforth will be used as designators for the domestic and foreign markets. Thirdly, we assume that there exists just one domestic equity security (with value-process \( S_{d,t} \)), just one foreign equity security (with value process \( S_{f,t} \)) and obviously just one exchange rate between the markets (with value process \( X_t \) [quoted as units of domestic currency per units of foreign currency]). Finally, we assume that \( S_d, S_f \) and \( X \) all follow Geometric Brownian Motion, and that the markets are complete. In formal terms, this translates to assuming the \( \mathbb{P} \) dynamics:

\[
\begin{align*}
    dX_t &= X_t \alpha_X dt + X_t \sigma_X dW_t, \\
    dS_{d,t} &= S_{d,t} \alpha_d dt + S_{d,t} \sigma_d^d dW_t, \\
    dS_{f,t} &= S_{f,t} \alpha_f dt + S_{f,t} \sigma_f^f dW_t, \\
    dB_{d,t} &= r_d B_{d,t} dt, \\
    dB_{f,t} &= r_f B_{f,t} dt,
\end{align*}
\]

where \( B \) is the risk free asset, \( dW_t \equiv (dW_{1,t}, dW_{2,t}, dW_{3,t}) \) is a column vector of three independent Wiener increments, \( \sigma \equiv (\sigma_1, \sigma_2, \sigma_3) \) is a three dimensional volatility column vector and \( \alpha \) and \( \sigma \) are constant and therefore adapted to the \( \{\mathcal{F}_t^W\}_{t \geq 0} \)-filtration \( \forall i \in \{X, d, f\} \).

Now, for comparison purposes, there is something fundamentally dissatisfactory with equation system (1) insofar as the SDEs of the domestic assets are quoted in the domestic
currency whilst the foreign assets are quoted in the foreign currency. To obtain a uniform domestic currency, we therefore really ought to consider the price processes \( S_{d,t} \equiv X_t S_{d,f,t} \) and \( B_{f,t} \equiv X_t B_{f,t} \). Applying Itô’s lemma to these definitions we obtain

\[
\begin{align*}
\frac{dS_{f,t}}{S_{f,t}} &= \frac{\partial S_{f,t}}{S_{f,t}}dt + \frac{\partial^2 S_{f,t}}{2S_{f,t}^2}d\tilde{W}_{f,t} + \frac{\partial^2 S_{f,t}}{2S_{d,f,t}^2}d\tilde{W}_{d,t} + \frac{\partial^2 S_{d,f,t}}{2S_{d,f,t}^2}d\tilde{W}_{d,t}, \\
\frac{dB_{f,t}}{B_{f,t}} &= \frac{\partial B_{f,t}}{B_{f,t}}dt + \frac{\partial^2 B_{f,t}}{2B_{f,t}^2}d\tilde{W}_{f,t}. 
\end{align*}
\]

Notice the the foreign bank account is not risk free from a domestic perspective for the obvious reason that the exchange rate between the currencies is stochastic.

1.2. The Hedging Portfolio. Consider now a generic contingent claim which depends on the exchange rate as well as the domestic and foreign equity securities. Specifically, we contemplate a derivative with a terminal pay-off function \( V_T = V(X_T, S_{d,T}, S_{f,T}) \). The fact that we here use the domestic form of the foreign stock bears little significance: we could equally well work with the state variables \((X_t, S_{d,t}, S_{f,t})\) given the one-to-one correspondence between \( S_f \) and \( \tilde{S}_f \). However, the choice of the former simplifies calculations somewhat when we set up the hedging portfolio of the value process \( V_t \).

The process of hedging currency derivatives is complicated somewhat by the existence of foreign currency which can be deposited in a foreign bank account offering, in a sense, a stochastic return (seen with the eyes of a foreign investor to that country). Specifically \( \tilde{B}_f \) diffuses as \( \tilde{B}_f \sigma_f \tilde{W}_t \) from the perspective of the home economy, but this is actually fortuitous as we now have a way to hedge exchange risk in the derivative. Thus, we posit a hedging portfolio which has positions in both the domestic stock, the foreign stocks and the foreign currency whilst the foreign assets are quoted in the foreign currency. To obtain a uniform domestic terms the value process of said portfolio is

\[
\Pi_t = B_{d,t} + \delta_d S_{d,t} + \delta_f \tilde{S}_{f,t} + \tilde{B}_{f,t}.
\]

The point is that incremental changes in the derivative value must be matched by incremental changes in the portfolio. To determine our exact portfolio position, we must therefore compute \( d\Pi_t \), which is done using Itô’s lemma:

\[
\begin{align*}
d\Pi_t &= \frac{\partial \Pi_t}{\partial t}dt + \frac{\partial \Pi_t}{\partial X_t}dX_t + \frac{\partial \Pi_t}{\partial S_{d,t}}dS_{d,t} + \frac{\partial \Pi_t}{\partial S_{f,t}}dS_{f,t} + \frac{1}{2} \left\{ \frac{\partial^2 \Pi_t}{\partial X_t^2}dX_t^2 + \frac{\partial^2 \Pi_t}{\partial S_{d,t}^2}dS_{d,t}^2 + \frac{\partial^2 \Pi_t}{\partial S_{f,t}^2}dS_{f,t}^2 \\
&\quad+ 2 \frac{\partial^2 \Pi_t}{\partial X_t \partial S_{d,t}}dX_t dS_{d,t} + 2 \frac{\partial^2 \Pi_t}{\partial X_t \partial S_{f,t}}dX_t dS_{f,t} + 2 \frac{\partial^2 \Pi_t}{\partial S_{d,t} \partial S_{f,t}}dS_{d,t} dS_{f,t} \right\}.
\end{align*}
\]

Inserting the appropriate dynamical equations \((1)\) and \((2)\) in the second order increments we obtain

\[
\begin{align*}
d\Pi_t &= \frac{\partial \Pi_t}{\partial t}dt + \frac{\partial \Pi_t}{\partial X_t}dX_t + \frac{\partial \Pi_t}{\partial S_{d,t}}dS_{d,t} + \frac{\partial \Pi_t}{\partial S_{f,t}}dS_{f,t} + \frac{1}{2} \left\{ X_t^2 \| \sigma_f \| \sigma_f \| \sigma_f \|^2 + S_{d,t}^2 \| \sigma_d \| \sigma_d \| \sigma_d \|^2 + S_{f,t}^2 \| \sigma_f \| \sigma_f \| \sigma_f \|^2 \right\}dt + \left\{ X_t S_{d,t} \sigma_f^\top \sigma_d \frac{\partial^2 V}{\partial S_{d,t} \partial \sigma_f} + \right. \\
&\left. + X_t \tilde{S}_{f,t} \sigma_f^\top \sigma_f \sigma_f \frac{\partial^2 V}{\partial S_{f,t} \partial \sigma_f} + S_{d,t} \tilde{S}_{f,t} \sigma_d^\top \sigma_d \frac{\partial^2 V}{\partial S_{d,t} \partial \sigma_f} \right\} dt,
\end{align*}
\]

where \( \| \sigma \| \equiv (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)^{1/2} \) is the Euclidian norm.
Comparing this with the self-financing condition
\[ d\Pi_t = dB_{d,t} + \delta_d dS_{d,t} + \delta_f \tilde{S}_{f,t} + dB_{f,t} = r_d dB_{d,t}dt + \delta_d dS_{d,t} + \delta_f \tilde{S}_{f,t} + r_f X_t B_{f,t} dt + B_{f,t} dX_t \]
we see that to successfully hedge the domestic and foreign equity risk we must set \( \delta = \partial V/\partial S_d \) and \( \delta_f = \partial V/\partial S_f \). More interestingly, to eliminate currency risk we must hold \( B_{f,t} = \partial V/\partial X \) units of (foreign) currency in the foreign bank account at time \( t \). Cancelling stochastic increments from both sides of \( dV_t = d\Pi_t \), and using the fact that \( B_{d,t} = V_t - \delta_d S_{d,t} - \delta_f \tilde{S}_{f,t} = \tilde{B}_{f,t} \) we deduce that that the currency derivative obeys the PDE
\[
\begin{align*}
\frac{\partial V}{\partial t} + X_t (r_d - r_f) \frac{\partial V}{\partial X} + S_{d,t} r_d \frac{\partial V}{\partial S_d} + \tilde{S}_{f,t} r_d \frac{\partial V}{\partial S_f} + \frac{1}{2} \left\{ X_t^2 ||\sigma_x||^2 \frac{\partial^2 V}{\partial x^2} 
+ S_{d,t}^2 ||\sigma_d||^2 + \tilde{S}_{f,t}^2 ||\sigma_f + \sigma_x||^2 \sigma_x \right\} \right) 
+ X_t \tilde{S}_{f,t} \sigma_x^T (\sigma_f + \sigma_x) \frac{\partial V}{\partial S_f} + S_{d,t} \tilde{S}_{f,t} \sigma_x^T (\sigma_f + \sigma_x) \frac{\partial V}{\partial S_d} 
\right\} = r_d V_t,
\end{align*}
\]
with the terminal condition \( V_T = V(X_T, S_{d,T}, \tilde{S}_{f,T}) \).

The solution to (4) is given by the Feynman-Kac formula:
\[
(5) \quad V_t = e^{-r_d (T-t)} E^Q \left[ V(X_T, S_{d,T}, \tilde{S}_{f,T}) \left| \mathcal{F}_t \right. \right] _{\mathcal{W}^Q}
\]
which can be seen as follows. Under the \( \mathcal{Q} \) dynamics:
\[
\begin{align*}
dS_{d,t} &= S_{d,t} \sigma_d^T dW_t^Q, \\
d\tilde{S}_{f,t} &= \tilde{S}_{f,t} \sigma_f^T dW_t^Q, \\
d\tilde{B}_{f,t} &= \tilde{B}_{f,t} \sigma_f^T dW_t^Q, \\
X_t &= X_t (r_d - r_f) dt + X_t \sigma_x^T dW_t^Q, \\
dS_{f,t} &= S_{f,t} (r_f - \sigma_f^T \sigma_f) dt + S_{f,t} \sigma_f^T dW_t^Q.
\end{align*}
\]
The first three are obvious (all tradable assets drift at the risk free rate) - the latter two follow by applying Itô’s lemma to \( X_t = \tilde{B}_{f,t}/B_{f,t} \) and \( S_{f,t} = \tilde{S}_{f,t}/X_t \). Thus, we can write out \( dZ_s \equiv d(e^{-r_d s} V_s) \) under the \( \mathcal{Q} \) measure with the aid of equation \( (3) \). Integrating \( dZ_s \) between \( t \) and \( T \) and using the governing PDE we obtain
\[
Z_T = Z_t + \int_t^T \left( S_{d,t} \sigma_d^T + \tilde{S}_{f,t} (\sigma_f^T + \sigma_x^T) + X_t \sigma_x^T \right) dW_s^Q.
\]
Assuming enough regularity (the integrands must be in \( L^2 \)) we can now apply the time \( t \) risk neutral expectation operator to this expression only to be left with \( (5) \) as desired.

Remark: The PDE \( (4) \) and the associated solution \( (5) \) are form invariant under the change of state variable \( S_f \rightarrow S_f \). Nonetheless, take care to notice that \( \delta_f = \partial V/\partial \tilde{S}_f = X_t^{-1} (\partial V/\partial S_f) \) which is important from a hedging perspective.
1.3. An Example: The Quanto Put. Per definitionem, the pay-off of a quanto put option is $Q_{P,T} = Y_0 \max \{ K - S_{T,T}, 0 \}$, where $Y_0$ is some (constant) exchange rate agreed upon in advance. To find the time $t$ value of this asset it is convenient to re-express the $Q$ dynamics of the foreign stock as

$$dS_{f,t} = S_{f,t}(r_d - (r_d - r_f + \sigma_f^\top \sigma_f))dt + S_{f,t}\sigma_f^\top dW_t^Q.$$  

This shows that the price process behaves as though a continuous dividend yield of $q = r_d - r_f + \sigma_f^\top \sigma_f$ is paid out. In other words, we can make appropriate substitutions in the standard Black Scholes formula for a dividend paying put options to obtain

(6) \[ Q_{P,t} = Y_0 e^{-r_d(T-t)} \left\{ K\Phi(-d_2) - e^{(r_f - \sigma_f^\top \sigma_f)(T-t)} S_{f,t}\Phi(-d_1) \right\} \]

where $\Phi$ is the cumulative standard normal distribution and

$$d_1 = \frac{\ln(S_{f,t}/K) + (r_f - \sigma_f^\top \sigma_f + \frac{1}{2}||\sigma_f||^2)(T-t)}{\sqrt{T-t}||\sigma_f||} \quad \text{and} \quad d_2 = d_1 - \sqrt{T-t}||\sigma_f||.$$  

The hedge position in the foreign stock, $\delta_f = X_t^{-1}\partial(QP)/\partial S_f$, is given by

$$\delta_f = X_t^{-1}Y_0 e^{-r_d(T-t)} \frac{\partial}{\partial S_f} \left\{ K(1 - \Phi(d_2)) - e^{(r_f - \sigma_f^\top \sigma_f)(T-t)} S_{f,t}(1 - \Phi(d_1)) \right\}$$

$$= X_t^{-1}Y_0 e^{-r_d(T-t)} \left\{ -K\phi(d_2)\frac{\partial d_2}{\partial S_f} + e^{(r_f - \sigma_f^\top \sigma_f)(T-t)} \left( S_{f,t}\phi(d_1)\frac{\partial d_1}{\partial S_f} + (\Phi(d_1) - 1) \right) \right\}$$

$$= X_t^{-1}Y_0 e^{-r_d(T-t)} \left\{ -K\phi(d_2) + e^{(r_f - \sigma_f^\top \sigma_f)(T-t)} S_{f,t}\phi(d_1) \frac{\partial d_1}{\partial S_f} \right\}$$

$$+ e^{(r_f - \sigma_f^\top \sigma_f)(T-t)}(\Phi(d_1) - 1)$$

In the first line we use the symmetry of the normal curve $\Phi(-x) = 1 - \Phi(x)$. In the second line we differentiate using the chain rule and introduce the notation $\phi$ for the standard normal pdf. The third line exploits the definition of $d_2$ in terms of $d_1$ and the final line the standard identity.

$$e^{-(r_d - r_f + \sigma_f^\top \sigma_f)(T-t)}S_{f,t}\phi(d_1) = Ke^{-r_d(T-t)}\phi(d_2).$$

It is now a straightforward exercise to compute the holding of currency in the foreign bank account, $B_{f,t} = \partial(QP)/\partial X$. In particular, since the quanto put only depends on the exchange rate through the state variable $S_f$:

$$\frac{\partial B_{P}}{\partial X} = \frac{\partial S_{f}}{\partial X} \frac{\partial Q_{P}}{\partial S_{f}} = -\frac{\partial S_{f}}{X} \frac{\partial Q_{P}}{\partial S_{f}} = -S_{f} \frac{\partial Q_{P}}{\partial S_{f}} = -S_{f} \delta_{f}.$$  

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1 Specifically, in a standard Black-Scholes environment, if $S_t$ is a stock price which follows Geometric Brownian Motion $dS_t = \mu S_t dt + \sigma S_t dW_t$ and which has a continuous dividend yield (meaning that the cumulative dividend follows $dD_t = q dt$), then the risk neutral representation of the stock dynamics is $dS_t = (r - q)S_t dt + \sigma S_t dW_t^Q$ cf. Björk, Arbitrage Theory in Continuous Time (third edition), p. 239.

2 Specifically, in the standard Black Scholes environment where our stock pays a continuous dividend yield, the relation is $Se^{-q(T-t)}\phi(d_1) = Ke^{-r(T-t)}\phi(d_2)$ as shown in lemma 3.15 of Stefanica, A Primer for the Mathematics of Financial Engineering (second edition), p.94.
Figure 1. **Hedging a Quanto Put** In this numerical simulation we show that when a quanto put option is hedged using the portfolio \( \Pi_t = B_t + \delta_f S_t X_t + X_t B_f \) where \( \delta_f = X_t^{-1} Y_0 e^{(r_f - \sigma_f \hat{\sigma}_f (T-t) \Phi(d_1) - 1)} \) and \( B_f = -S_f \delta_f \) then the standard deviation of the hedge error goes to zero as we increase the number of (discrete) hedges per year. The parameter specifications used for the simulation are: \( r_d = 0.05 \), \( r_f = 0 \), \( \sigma_X = (0.05, 0.05) \), \( \sigma_f = (0.2) \), \( S_f = 100 \), \( K = 100 \), \( X_0 = Y_0 = 0.01 \). For each plotted data point 5000 Monte Carlo stock price paths were generated.

2. **Erroneous Hedges: The Fundamental Theorem of Derivative Trading for Currency Options**

As always, when setting up a hedging portfolio of a derivative, the financial practitioner is haunted by the fact that there is no way to read off the true volatilities of the underlyings. For a currency derivative of the form \( V_t = V(X_t, S_{d,t}, \tilde{S}_{f,t}) \) this problem translates to specifying a total of nine parameters in the volatility matrix

\[
\Sigma = \begin{pmatrix} \sigma_X & \sigma_D & \sigma_F \\ \sigma_D & \sigma_{d1} & \sigma_{d2} \\ \sigma_F & \sigma_{f1} & \sigma_{f2} \end{pmatrix}
\]

when doing an actual hedge. Specifically, imagine that we have acquired a currency derivative at time \( t_0 \) for a price \( V_{t_0}^i \) where the superscript \( i \) represents the market view of the volatility matrix \( \Sigma \) (obviously, these implied values are underdetermined). If we decide to hedge our exposure to fluctuations in the equity securities and the exchange rate, using what we might call our own arbitrary estimate of volatility matrix, the value process of the hedged portfolio will be of the form

\[
\Pi_{t/a} = V_t^i + B_{d,t} - \frac{\partial V^a}{\partial S_d} S_{d,t} - \frac{\partial V^a}{\partial S_f} \tilde{S}_{f,t} + \tilde{B}_{f,t}^a
\]

where the superscript \( a \) represents the arbitrary assumption \( \Sigma = \Sigma^a \). As always, \( \tilde{B}_{f,t}^a = X_t B_{f,t}^a \) where \( B_{f,t}^a = -\partial V^a / \partial X \) is the instantaneous holding in the foreign account and the domestic account \( B_{d,t} \) is adjusted continuously over the lifetime of the option \([t_0, T]\).
such that the net portfolio value is nil. Nonetheless, as we shall shortly see, our misspecified hedge will cause the portfolio to bleed, meaning that we will incur a net profit or loss by holding \( \Pi_i^{\prime/a} \) over any interval of time of non-zero extension. An exact formula for the magnitude of this bleeding can be derived as a generalisation of what is sometimes called the Fundamental Theorem of Derivative Trading.

Till this end, consider the incremental change in the portfolio value

\[
d\Pi_t^{\prime/a} = dV_t^i + dB_{d,t} - \frac{\partial V^a}{\partial S_d} dS_{d,t} - \frac{\partial V^a}{\partial S_f} d\tilde{S}_{f,t} + B_{f,t}^a dX_t^i + X_t dB_{f,t}^a
\]

where \( r \) represents the real changes i.e. the increments brought about by the volatility matrix \( \Sigma^r \) which in reality should go into the asset dynamics (and of which \( \Sigma^a \) and \( \Sigma^f \) are approximations). Since \( dB_{i,t} = r_i B_{i,t} dt \) for \( i \in \{ d, f \} \) and \( B_{d,t} \) is chosen such that \( \Pi_t^{\prime/a} = 0 \) we have that

\[
d\Pi_t^{\prime/a} = dV_t^i + r_d \left( \frac{\partial V^a}{\partial S_d} S_{d,t} + \frac{\partial V^a}{\partial S_f} \tilde{S}_{f,t} - \frac{\partial V^a}{\partial X} X_t - V_t^i \right) dt \\
- \frac{\partial V^a}{\partial S_d} dS_{d,t} - \frac{\partial V^a}{\partial S_f} d\tilde{S}_{f,t} - \frac{\partial V^a}{\partial X} dX_t^i - r_d X_t \frac{\partial V^a}{\partial X} dt
\]

(7)

Consider now the change \( dV_t^a \). From Itô’s lemma

\[
dV_t^a = \frac{\partial V^a_t}{\partial t} dt + \frac{\partial V^a_t}{\partial S_d} dS_{d,t} + \frac{\partial V^a_t}{\partial S_f} d\tilde{S}_{f,t} + \frac{1}{2} \left\{ S_{d,t}^2 \frac{\partial^2 V^a_t}{\partial S_d^2} + S_{f,t}^2 \frac{\partial^2 V^a_t}{\partial S_f^2} + \frac{1}{2} \left( X_t^2 \left| \sigma_X^a \right|^2 - \left| \sigma_X^a \right|^2 \right) \frac{\partial^2 V^a_t}{\partial X^2} \right\} dt + \left\{ X_t S_{d,t} \sigma_X^a \sigma_d^a \frac{\partial^2 V^a_t}{\partial S_d \partial X} + X_t \tilde{S}_{f,t} \sigma_X^a \sigma_f^a \frac{\partial^2 V^a_t}{\partial S_f \partial X} + X_t \tilde{S}_{f,t} \sigma_f^a \sigma_f^a \frac{\partial^2 V^a_t}{\partial S_d \partial S_f} + S_{d,t} \tilde{S}_{f,t} \sigma_f^a \sigma_f^a \frac{\partial^2 V^a_t}{\partial S_f \partial S_f} \right\} dt,
\]

where we have inserted explicit expressions for the real dynamics in the second order increments. Per definition \( V_t^a \) satisfies a PDE of the form (4) where \( \sigma_X = \sigma_X^a, \sigma_d = \sigma_d^a \) and \( \sigma_f = \sigma_f^a \). Inserting this in place of \( \partial V^a_t / \partial t \):

\[
dV_t^a = r_d V_t^a dt + \frac{\partial V^a_t}{\partial S_d} (dS_{d,t} - r_d S_{d,t} dt) + \frac{\partial V^a_t}{\partial S_f} (d\tilde{S}_{f,t} - r_d \tilde{S}_{f,t} dt) + \frac{1}{2} \left\{ S_{d,t}^2 \left| \sigma_d^a \right|^2 - \left| \sigma_d^a \right|^2 \right\} \frac{\partial^2 V^a_t}{\partial S_d^2} dt + \left\{ X_t S_{d,t} \sigma_d^a \sigma_d^a \frac{\partial^2 V^a_t}{\partial S_d \partial S_d} + X_t \tilde{S}_{f,t} \sigma_f^a \sigma_f^a \frac{\partial^2 V^a_t}{\partial S_f \partial S_f} + X_t \tilde{S}_{f,t} \sigma_f^a \sigma_f^a \frac{\partial^2 V^a_t}{\partial S_d \partial S_f} + S_{d,t} \tilde{S}_{f,t} \sigma_f^a \sigma_f^a \frac{\partial^2 V^a_t}{\partial S_f \partial S_f} \right\} dt.
\]

We can write this more succinctly as
\[
0 = -dV_i^n + r_dV_i^n dt + \frac{\partial V_i^n}{\partial X} (dX_i^n - X_i(r_d - r_f)dt) + \frac{\partial V_i^n}{\partial S_d^n} (dS_{d,t}^n - r_dS_{d,t}^n dt) \]
(8)
\[
+ \frac{\partial V_i^n}{\partial S_f^n} (dS_{f,t}^n - r_dS_{f,t}^n dt) + \frac{1}{2} \text{tr}[(\Lambda' \Lambda'^T - \Lambda^a \Lambda'^a) D^a]
\]
where \(\text{tr}\) is the trace operator, \(\Lambda \equiv (\sigma_X^T, \sigma_d^T, \sigma_f^T + \sigma_X^T)\) and \(D^a = C \circ H^a\) is the Hadamard product between the matrix of squared state variables \(C \equiv XX^T\) (where \(X \equiv (X_i, S_{d,t}, S_{f,t})\)) and the Hessian:
\[
H^a = \begin{pmatrix}
\frac{\partial^2 V^a}{\partial x^2} & \frac{\partial^2 V^a}{\partial x \partial s_d} & \frac{\partial^2 V^a}{\partial x \partial s_f} \\
\frac{\partial^2 V^a}{\partial x \partial s_d} & \frac{\partial^2 V^a}{\partial s_d^2} & \frac{\partial^2 V^a}{\partial s_d \partial s_f} \\
\frac{\partial^2 V^a}{\partial x \partial s_f} & \frac{\partial^2 V^a}{\partial s_d \partial s_f} & \frac{\partial^2 V^a}{\partial s_f^2}
\end{pmatrix}.
\]

Inserting the zero term (8) in (7) we finally obtain after some rewriting
\[
d\Pi^{i/a}_t = e^{r_d t} d(e^{-r_d t} (V_i^n - V_i^a)) + \frac{1}{2} \text{tr}[(\Lambda' \Lambda'^T - \Lambda^a \Lambda'^a) D^a].
\]
Hence the time \(t_0\) present-valued portfolio increment is
\[
dPV_{t_0}[\Pi^{i/a}_T] = e^{r_d t_0} d(e^{-r_d t} (V_i^n - V_i^a)) + \frac{1}{2} e^{-r_d (t-t_0)} \text{tr}[(\Lambda' \Lambda'^T - \Lambda^a \Lambda'^a) D^a]
\]
So over the entire lifetime of the option the profit-and-loss has amounted to
\[
PV_{t_0}[\Pi^{i/a}_T] = V_i^n - V_i^a + \frac{1}{2} \int_{t_0}^T e^{-r_d (t-t_0)} \text{tr}[(\Lambda' \Lambda'^T - \Lambda^a \Lambda'^a) D^a] dt,
\]
where we have made use of the fact that options at expiry are independent of volatility.